Approximation and Online Algorithms with Applications

#8

~~Multi-Armed Bandit~~ Weight Majority Algorithm

- We are focusing on 1 stock, and thinking if we should buy this stock at time $1, \dots, T$.
- For each time t, we are going to hear opinion from n experts. $x_1^{(t)}, x_2^{(t)}, \dots, x_n^{(t)}$, and make a decision.
  $\in \{$buy, not-buy$\}$
- We will immediately know the result of the decision immediately.
  [Stock get up (should buy) or Stock get down (should not buy)]

Input: $x_1^{(t)}, x_2^{(t)}, \dots, x_n^{(t)} \in \{\text{buy}, \text{not-buy}\}$ for all t

Output: $y^{(t)} \in \{\text{buy}, \text{not-buy}\}$ for all t.

Objective Function:

$$p^{(t)} = \begin{cases} 1 & \text{if } y^{(t)} \text{ is the right decision} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Maximize. } \sum_{t=1}^{T} p^{(t)}.$$

Key Idea: We should believe in experts who give us a good recommendation in previous step.

Algorithm

1: Weight for all experts: $w_1, \dots, w_n = 1$

2: For $t = 1$ to T:

Sum of weight of experts who suggest buy

If $\sum_{i:x_i^{(t)}=\text{buy}} w_i \geq \sum_{i:x_i^{(t)}=\text{not-buy}} w_i$:

Sum of weight of experts who suggest not buy

$y^{(t)} = \text{buy}$

else

$y^{(t)} = \text{not-buy}$.

For all expert i that makes a wrong suggestion, $w_i \leftarrow w_i/2$.

We believe them less and put a smaller to them.

Example

Time	Expert 1	Expert 2	Expert 3	
1	Buy 1	Not-Buy 1	Not-Buy 1	→ Not-Buy (wrong)
2	Not-Buy 1	Buy 1/2	Not-Buy 1/2	→ Not-Buy (right)
3.	Buy 1	Not-Buy 1/2	Not-Buy 1/4	→ Buy (right)

2 right decisions

Score = 2

Theorem: # mistake decisions $\leq 2.41(m + \log_2 n)$

where m is # mistake of best expert.

Proof: Weight of the best expert after the T^{th} iteration

$= \frac{1}{2^m}$. (The weight is initialized to 1, and is divided by 2 for m times)

The sum of all weights after the T^{th} iteration $= W_T$

$\geq \frac{1}{2^m}$.

The sum of all weight after the t^{th} iteration W_t

$=$ Sum of weight with correct suggest C_t $+$ $\boxed{\text{Sum of weight with incorrect suggest } I_t}$

larger when we make a mistake decision.

$$W_{t+1} = C_t + I_t/2$$

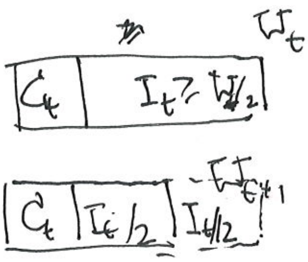
$$= W_t - I_t/2$$

$$\leq W_t - W_t/4$$

$$= 3/4\, W_t.$$

$I_t \geq C_t$

$\cancel{I_t \geq \frac{1}{2} W_t}$



$I_t/2 \geq W/4$

$W_{t+1} \leq 3/4\, W_t$

(when we make a mistake decision)

$W_1 = n$ [The weights of all n experts are 1]

$W_{P_1} \leq 3/4\, n$ [After we made 1 mistake]

$W_{P_2} \leq \cancel{(3/4)^2}\; 3/4\, W_{P_1} \leq (3/4)^2 n$ [After we made 2 mistakes]

$W_{P_3} \leq (3/4)^3 n$ [After we made 3 mistakes]

$\vdots$

$W_T \leq (3/4)^M n$ [After M mistakes]

$W_T \geq \frac{1}{2^m}$

$$\frac{1}{2^m} \leq (3/4)^M n$$

$$-m = \lg 2^{-m} = \lg \frac{1}{2^m} \leq \lg\left[(3/4)^M n\right]$$

$$= \lg (3/4)^M + \lg n$$

$$= M \lg 3/4 + \lg n.$$

$$-m \leq (-0.415)\, M + \lg n$$

$$0.415\, M \leq m + \lg n$$

$$M \leq 2.41\,[m + \lg n]$$

$\boxed{\mathbb{Z}}$

Use randomness

1: Weight for all expert : $w_1, \cdots, w_n = 1$

2: For $t = 1$ to T:

$W_{buy} = \sum_{i : x_i^{(t)} = buy} w_i$

$W_{no\text{-}buy} = \sum_{i : x_i^{(t)} = not\text{-}buy} w_i$

$P_{buy} = W_{buy} / (W_{buy} + W_{no\text{-}buy})$

$P_{not\text{-}buy} = 1 - P_{buy}$

Select $y^{(t)}$ according to that probability

For all expert i that makes a wrong suggestion, $w_i \leftarrow w_i / 2$

Theorem Expected [# mistake decisions] $\leq$ $1.39\, m + 2 \lg n$

decrease from $2.41\, m + 2.42 \lg n$.

Proof

$F^{(t)}$:= Weight for the wrong decision

→ Prob. that we make a mistake decision.

$$\text{Expected [\# mistake decision]} = \sum_{t=1}^{T} F^{(t)}$$

$$W^{(t)} := \sum_{i=1}^{T} w_i \text{ (at time } t)$$

$$W^{(1)} = \sum_{i=1}^{n} 1 = n.$$

Correct decision

wrong decision

$$W^{(2)} = n \cdot (1 - F^{(1)}) + n \cdot F^{(1)}/2$$

$$= n(1 - F^{(1)}/2)$$

~~Incorrect~~ correct prediction

incorrect prediction divide by 2

$$\leq \boxed{n(1 - F^{(1)}/2) \cdot F^{(2)}}$$

$$\boxed{n(1 - F^{(1)}/2) \cdot (1 - F^{(2)})}$$

~~divide by 2~~.

$$\bar{W}^{(3)} = n(1-F^{(1)}/2)(1-F^{(2)}) + n(1-F^{(1)}/2)\,F^{(2)}/2$$

$$= n(1-F^{(1)}/2)(1-F^{(2)}/2)$$

$$\vdots$$

$$\bar{W}^{(T+1)} = n(1-F^{(1)}/2)(1-F^{(2)}/2)\cdots(1-F^{(T)}/2)$$

$$= n\prod_{t=1}^{T}(1-F^{(t)}/2)$$

Weight of the best expert

$$= \left(\frac{1}{2}\right)^m$$

$1+x \le e^x$

$1-x \le e^{-x}$

$\ln\cancel{\lg}(1-x) \le -x$

$$n\prod_{t=1}^{T}(1-F^{(t)}/2) \geq \left(\frac{1}{2}\right)^m$$

$$\overset{\ln}{\cancel{\lg}}\, n + \sum_{t=1}^{T}\overset{\ln}{\cancel{\lg}}(1-F^{(t)}/2) \geq \overset{\ln}{\cancel{\lg}}\left(\frac{1}{2}\right)^m$$

$$\overset{\ln}{\cancel{\lg}}\, n + \sum_{t=1}^{T}(-F^{(t)}/2) \geq m\overset{\ln}{\cancel{\lg}}\left(\frac{1}{2}\right)$$

-0.70

$\cancel{-1.39}$

$$-\overset{\ln}{\cancel{\lg}}\, n + -\sum_{t=1}^{T}(-F^{(t)}/2) \leq -m\,\overset{\ln}{\cancel{\lg}}\left(\frac{1}{2}\right) = \cancel{m\lg 2}$$

$$\sum_{t=1}^{T}F^{(t)}/2 \leq \ln n + 0.7m$$

$$\sum_{t=1}^{T}F^{(t)} \leq 2\ln n + 1.4m \qquad \text{//}$$